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KINETICS OF MASS TRANSFER IN A CAPILLARY WITH EVAPORATION FROM THE INTERNAL SURFACE

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Abstract—The paper describes mathematical formulation and the method of solution of the problem on mass-transfer kinetics in a capillary, involving evaporation from the inner surface. A linearized BGK model equation is solved, the distribution function being chosen of such a form that the boundary conditions could be satisfied as well as the original equation divided into two equations for the functions depending on z and r, respectively. Expressions are obtained for distributions of macroscopic quantities characterizing the flow of vapor. In the second part of the paper, on the basis of the given general technique, mass transfer in a capillary is studied analytically for limiting cases of the continuous medium and free-molecular regime. The table and figures present the values of quantities which allow the cross-section averaged flow to be determined in an axial direction as well as the flow on a lateral surface, and the density and velocity profiles for different rarefaction regimes.

NOMENCLATURE

p, pressure;
Z, longitudinal coordinate;
$\mathbf{R}$, $= (X, Y)$, transverse coordinates;
f', function of molecules distribution by velocities;
τ', relaxation time;
m, mass of a molecule;
k, Boltzmann constant;
$\boldsymbol{\xi}$, molecular velocity;
Kn, Knudsen number;
T, temperature;
v_i, nondimensional macroscopic vapor velocity components;
n_s, density of saturated vapor;
$\mathbf{n}$, unit vector of the normal to the lateral surface.

INTRODUCTION

KNOWLEDGE of mass transfer in a gas phase inside porous bodies is important for various technological processes such as heterogeneous catalysis, drying etc. In the majority of cases, calculation of these is based on some limiting relationships corresponding to either the hydrodynamic ($Kn \to 0$) or the free molecular flow regimes in pores ($Kn \to \infty$) [1, 2]. It is quite natural, that to completely define the structure of the flow and the net flow of substance from pores with regard for rarefaction, as well as to specify the limits of application of these or other phenomenological relationships, it is necessary to solve a kinetic equation for a flow in a separate capillary with account for physical and chemical transitions at the walls. Besides, the solution for the continuum regime, which is difficult to obtain directly from the Navier–Stokes equations, is obtained here as a limiting solution ($Kn \to 0$) from the equations for the appropriate moments of the distribution function.

This kind of inner boundary problems in the kinetic theory of gases is less studied than the problems of outer flow. Moreover, consideration of phase or chemical transitions at the walls of a pore essentially complicates the kinetic problem as compared with the Poiseuille flow analyzed in [3, 4].

One of the first investigations in this field is the work [5] which deals with the kinetic theory of mass transfer involving catalysis in a cylindrical channel, but it concerns a particular case of trace gas diffusion in the main gas, i.e. at negligible mass velocity.

This paper considers vapor flow in a semi-infinite ($Z < 0$) cylindrical capillary of the radius r_0, at the open end of which, $Z = 0$, a constant pressure p_0 is maintained, which is less than the saturated vapor pressure $p_s(T)$. Here evaporation occurs over the entire inner capillary surface having a constant temperature. The difference $p_s(T) - p_0$ is assumed to be small, therefore the vapor flow is slow too.

The first part of the paper deals with formulation of the problem and the general method of solution, and the second gives a detailed description of the limiting vapor flow regimes, and presents the calculation results for a wide range of Knudsen numbers.

I

The analysis of the vapor flow is based on the BGK model kinetic equation [4, 6]

$$\boldsymbol{\xi}_R \frac{\partial f'}{\partial \mathbf{R}} + \xi_Z \frac{\partial f'}{\partial Z} = \frac{f_0' - f'}{\tau'}, \qquad (1)$$

where

$$\mathbf{R} = (X, Y),$$

$$f_0' = n'\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left\{-\frac{m(\boldsymbol{\xi} - \mathbf{v}')^2}{2kT}\right\}$$

is the Maxwellian distribution function.

Introducing the dimensionless coordinates and velocities

$$z = \frac{Z}{r_0}, \quad \mathbf{r} = \frac{\mathbf{R}}{r_0}, \quad \mathbf{u} = \boldsymbol{\xi}\left(\frac{m}{2kT}\right)^{1/2}$$

reduces equation (1) to the form

$$\mathbf{u}_r \frac{\partial F}{\partial \mathbf{r}} + u_z \frac{\partial F}{\partial z} = \frac{F_0 - F}{\tau}, \qquad (2)$$

where

$$F = \frac{f'}{n_s}\left(\frac{2kT}{m}\right)^{3/2}, \quad \tau = \frac{\tau'}{r_0}\left(\frac{2kT}{m}\right)^{1/2} \approx 2Kn.$$

Let us formulate the boundary conditions for equation (2). Assuming the coefficient of evaporation equal to unity gives for the particles leaving the capillary wall

$$F(\mathbf{n}\cdot\mathbf{u}_r < 0) = \frac{\exp\{-u^2\}}{\pi^{3/2}}. \qquad (3)$$

The density of particles at $z = 0$ is

$$n_0 = \frac{p_0}{kT}. \qquad (4)$$

The solution to the problem (2)–(4) is sought in the form

$$F = \frac{\tilde{n}(z)}{\pi^{3/2}} \exp(-u^2)[1 + \tilde{V}(z)\phi(\mathbf{u}, r)] \qquad (5)$$

i.e. it reduces to determination of the functions $\tilde{n}$, $\tilde{V}$ and ϕ. By definition

$$nv_i = \int_{-\infty}^{\infty} F u_i \,\mathrm{d}^3u; \quad i = x, y, z \quad \left(\text{here } n = \frac{n'}{n_s}\right).$$

Let us introduce the following notations

$$g = \int_{-\infty}^{\infty} \exp(-u^2)\phi \,\mathrm{d}^3u,$$

$$f_i = \int_{-\infty}^{\infty} \exp(-u^2)\phi u_i \,\mathrm{d}^3u.$$

Having linearized the equilibrium distribution function, we calculate the RHS of equation (2)

$$F_0 - F = \tilde{n}(z)\,\tilde{V}(z)\frac{\exp(-u^2)}{\pi^{3/2}} \times (g + 2\mathbf{u}\cdot\mathbf{f} - \pi^{3/2}\phi). \qquad (6)$$

At $r = 1$ equations (3) and (5) yield

$$\phi(\mathbf{u}_r \cdot \mathbf{n} < 0) = \left[\frac{1}{\tilde{n}(z)} - 1\right]\frac{1}{\tilde{V}(z)}. \qquad (7)$$

This quantity should be constant, since the function ϕ at $r = 1$ depends only on the molecular velocity, while the RHS, only on z. Hence, the following relationship holds

$$\left[\frac{1}{\tilde{n}(z)} - 1\right]\frac{1}{\tilde{V}(z)} = -C. \qquad (8)$$

Substitution of expressions (5), (6) and (8) into (2) gives

$$\tau u_x \frac{\partial \phi}{\partial x} + \tau u_y \frac{\partial \phi}{\partial y} + \phi + \tau u_z(\phi + C)\frac{\mathrm{d}}{\mathrm{d}z} \times \{\ln[\tilde{n}(z)\,\tilde{V}(z)]\} = \frac{1}{\pi^{3/2}}(g + 2\mathbf{u}\cdot\mathbf{f}). \qquad (9)$$

We may show that $(\mathrm{d}/\mathrm{d}z)\{\ln[\tilde{n}(z)\,\tilde{V}(z)]\}$ is independent of z. Actually, the continuity equation for a gas with account of (5) has the form

$$\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + f_z \frac{\mathrm{d}}{\mathrm{d}z}\{\ln[\tilde{n}(z)\,\tilde{V}(z)]\} = 0. \qquad (10)$$

Since f_x, f_y, f_z are the functions of r, equation (10) yields

$$\frac{\mathrm{d}}{\mathrm{d}z}\ln[\tilde{n}(z)\,\tilde{V}(z)] = \beta(\tau) \qquad (11)$$

and equation (9) takes the form

$$\tau u_x \frac{\partial \phi}{\partial x} + \tau u_y \frac{\partial \phi}{\partial y} + \phi + \tau u_z \beta(\phi + C) = \frac{1}{\pi^{3/2}}(g + 2\mathbf{u}\cdot\mathbf{f}). \qquad (12)$$

Thus, the form of expression (5) for the function F has made it possible to subdivide the original equation into equations (11) and (12) for the functions depending on z and $\mathbf{r}$, respectively. It should be emphasized that (12) differs essentially from the similar equations for the case of impermeable walls [3,4, 7] in that the last term on the LHS of (12) is not constant but depends on the desired function ϕ.

Let us write the solution to equation (12) in an integral form (integration is performed along the characteristics of the equation) taking account of the boundary condition (7) and relation (8)

$$\phi = -C\exp\left(-\frac{b}{\tau u_r}\right) + \frac{1}{\pi^{3/2}\tau}\int_0^b \frac{\exp\left(-\frac{s}{\tau u_r}\right)}{u_r}(g + 2\mathbf{u}\cdot\mathbf{f})_{r'}\,\mathrm{d}s - \beta\int_0^b \frac{u_z}{u_r}\exp\left(-\frac{s}{\tau u_r}\right)(\phi|_{r'} + C)\,\mathrm{d}s. \qquad (13)$$

Here

$$u_r = \sqrt{(u_x^2 + u_y^2)}, \quad b = r\cos\theta + \sqrt{(1 - r^2\sin^2\theta)}$$

is the segment of the characteristic from the capillary wall to the point r (Fig. 1); the characteristics of equation (12) are straight lines, i.e.

$$x - x' = s\cos\theta, \quad y - y' = s\sin\theta, \quad \cos\theta = \frac{u_x}{u_r},$$

$$\sin\theta = \frac{u_y}{u_r}, \quad s = \sqrt{[(x - x')^2 + (y - y')^2]}.$$

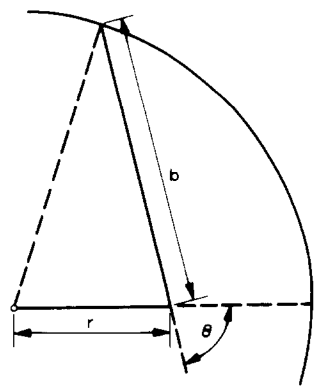

FIG. 1. Geometry of the problem.

Multiplying the both sides of equation (13) successively by $\exp(-u^2)u_i$ and $\exp(-u^2)$ and integrating with respect to the space of velocities $\mathbf{u}$, we obtain a set of equations:

$$
\begin{aligned}
f_z = {} & \frac{1}{\pi^{3/2}\tau}\int_{-\infty}^{\infty}\int_0^b \frac{\exp\left(-\frac{s}{\tau u_r}-u^2\right)}{u_r} f_z(r')\,\mathrm{d}s\,\mathrm{d}^3u \\
& -\beta\int_{-\infty}^{\infty}\int_0^b \frac{u_z^2}{u_r}\exp\left(-\frac{s}{\tau u_r}-u^2\right) \\
& \times(\phi|_{r'}+C)\,\mathrm{d}s\,\mathrm{d}^3u, \quad (14)
\end{aligned}
$$

$$
\begin{aligned}
f_j = {} & -\pi^{1/2}C\int_{-\infty}^{\infty} u_j\exp\left(-\frac{b}{\tau u_r}-u_r^2\right)\mathrm{d}^2u_r \\
& +\frac{1}{\pi^{3/2}\tau}\int_{-\infty}^{\infty}\int_0^b \frac{u_j}{u_r}\exp\left(-\frac{s}{\tau u_r}-u^2\right) \\
& \times(g+2\mathbf{u}\cdot\mathbf{f})_{r'}\,\mathrm{d}s\,\mathrm{d}^3u \\
& -\beta\int_{-\infty}^{\infty}\int_0^b \frac{u_zu_j}{u_r}\exp\left(-\frac{s}{\tau u_r}-u^2\right)\phi|_{r'}\,\mathrm{d}s\,\mathrm{d}^3u \\
& (j=x,y), \quad (15)
\end{aligned}
$$

$$
\begin{aligned}
g = {} & -\pi^{1/2}C\int_{-\infty}^{\infty}\exp\left(-\frac{b}{\tau u_r}-u_r^2\right)\mathrm{d}^2u_r \\
& +\frac{1}{\pi^{3/2}\tau}\int_{-\infty}^{\infty}\int_0^b \frac{\exp\left(-\frac{s}{\tau u_r}-u^2\right)}{u_r} \\
& \times(g+2\mathbf{u}\cdot\mathbf{f})_{r'}\,\mathrm{d}s\,\mathrm{d}^3u \\
& -\beta\int_{-\infty}^{\infty}\int_0^b \frac{u_z}{u_r}\exp\left(-\frac{s}{\tau u_r}-u^2\right)\phi|_{r'}\,\mathrm{d}s\,\mathrm{d}^3u. \quad (16)
\end{aligned}
$$

For the cross-section-averaged density of particles at the exit we have

$$
\begin{aligned}
n_0 &= \frac{1}{\Sigma}\int_\Sigma\int_{-\infty}^{\infty} F\,\mathrm{d}^3u\,\mathrm{d}\Sigma \\
&= 1+\tilde{n}(0)\,\tilde{V}(0)\left(C+\frac{2}{\pi^{3/2}}\int_0^1 gr\,\mathrm{d}r\right) \quad (17)
\end{aligned}
$$

where Σ is the cylinder cross-section area. Density and flows of particles in axial and radial directions following (5), (8), (11) and (17) are expressed as

$$
\begin{aligned}
n(z,r) &= 1+(n_0-1)\exp(\beta z)\frac{\kappa(r)}{2\int_0^1 \kappa r\mathrm{d}r}, \\
I_i(z,r) &= nv_i = (n_0-1)\exp(\beta z)\frac{f_i(r)}{2\pi^{3/2}\int_0^1 \kappa r\,\mathrm{d}r}
\end{aligned} \quad (18)
$$

where

$$\kappa = C+\frac{g}{\pi^{3/2}}.$$

Thus, to determine the density and flows, it is necessary to calculate the functions $f_i(r)$ and $\kappa(r)$.

The analysis of the integrals on the RHS of equation (13) shows that at not very large τ integration in the vicinity of the point $s=0$ makes the main contribution to the values of integrals. On this basis, expanding the functions g, f_i and ϕ into the Taylor series in the vicinity of the point $s=0(x=x', y=y')$ and restricting ourselves to the terms of order s^2, we pass from the integral equation (13) to an integro-differential equation. The expansion of integrands into the Taylor series is performed following [7], where evaporation only from the bottom of the capillary is taken into account. The resulting equation is rather cumbersome. It can be simplified by introducing the following notation

$$
\begin{aligned}
& b\exp\left(-\frac{b}{\tau u_r}\right)+\tau u_r\left[\exp\left(-\frac{b}{\tau u_r}\right)-1\right]=M, \\
& b^2\exp\left(-\frac{b}{\tau u_r}\right)+2\tau u_r\left\{b\exp\left(-\frac{b}{\tau u_r}\right)\right. \\
& \quad \left.+\tau u_r\left[\exp\left(-\frac{b}{\tau u_r}\right)-1\right]\right\}=N, \\
& \frac{1}{\pi^{3/2}}(g+2xfu_r\cos\theta+2yfu_r\sin\theta+2u_zf_z)=Q, \\
& \phi+\tau u_z\beta(\phi+C)-Q=G, \\
& \frac{\partial Q}{\partial x}\cos\theta+\frac{\partial Q}{\partial y}\sin\theta=\bar{L}(Q), \\
& \frac{\partial^2 Q}{\partial x^2}\cos^2\theta+2\frac{\partial^2 Q}{\partial x\partial y}\sin\theta\cos\theta+\frac{\partial^2 Q}{\partial y^2}\sin^2\theta=\bar{P}(Q).
\end{aligned} \quad (19)
$$

Represent the functions f_x and f_y in the form

$$f_x = xf(r), \quad f_y = yf(r). \quad (20)$$

Taking into account (19), (20) and using the expression obtained as a result of differentiation of equation (12) with respect to x and y, we reduce the above integro-differential equation to the form

$$
\begin{aligned}
\phi = {} & -C\exp\left(-\frac{b}{\tau u_r}\right)+\left[\exp\left(-\frac{b}{\tau u_r}\right)-1\right](G-\phi) \\
& +M\left[\bar{L}(Q)+\beta\frac{u_z}{u_r}G\right] \\
& +\frac{N}{2}\frac{1}{\tau u_r^2}\beta u_z[G(1+\tau\beta u_z)+\tau u_r\bar{L}(Q)] \\
& -\frac{N}{2}\bar{P}(Q). \quad (21)
\end{aligned}
$$

Let us consider equation (14). Since the kernels of the integrals entering into the RHS of the above equation have singularity at the point $s=0$ [7], we expand the integrand in the vicinity of the singularity point, restricting ourselves to the terms of order s^2:

$$
\begin{aligned}
& \int_{-\infty}^{\infty}\int_0^b \frac{\exp\left(-\frac{s}{\tau u_r}-u^2\right)}{u_r} f_z(r')\,\mathrm{d}s\,\mathrm{d}^3u \\
& = \pi^{1/2}\int_{-\infty}^{\infty}\int_0^b \frac{\exp\left(-\frac{s}{\tau u_r}-u_r^2\right)}{u_r} \\
& \quad \times[f_z(r)-\bar{L}(f_z)s+\tfrac{1}{2}\bar{P}(f_z)s^2]\,\mathrm{d}s\,\mathrm{d}^2u_r
\end{aligned}
$$

$$= -\pi^{1/2}\tau\Big\{f_z\Big[\int_0^{2\pi} T_1\Big(\frac{b}{\tau}\Big)\mathrm{d}\theta-\pi\Big]$$
$$-\frac{\mathrm{d}f_z}{\mathrm{d}r}\int_0^{2\pi}\Big[bT_1\Big(\frac{b}{\tau}\Big)+\tau T_2\Big(\frac{b}{\tau}\Big)\Big]\cos\theta\,\mathrm{d}\theta$$
$$+\tfrac{1}{2}\frac{\mathrm{d}^2f_z}{\mathrm{d}r^2}\int_0^{2\pi}\cos^2\theta\Big[b^2T_1\Big(\frac{b}{\tau}\Big)+2\tau bT_2\Big(\frac{b}{\tau}\Big)$$
$$+2\tau^2T_3\Big(\frac{b}{\tau}\Big)-\tau^2\Big]\mathrm{d}\theta$$
$$+\frac{1}{2r}\frac{\mathrm{d}f_z}{\mathrm{d}r}\int_0^{2\pi}\sin^2\theta\Big[b^2T_1\Big(\frac{b}{\tau}\Big)+2\tau bT_2\Big(\frac{b}{\tau}\Big)$$
$$+2\tau^2T_3\Big(\frac{b}{\tau}\Big)-\tau^2\Big]\mathrm{d}\theta\Big\},$$

where

$$T_n(y)=\int_0^{\infty}x^n\exp\Big(-x^2-\frac{y}{x}\Big)\mathrm{d}x\,[8].$$

Now equation (21) is used to calculate the second integral on the RHS of equation (14):

$$\int_{-\infty}^{\infty}\int_0^b\frac{u_z^2}{u_r}\exp\Big(-\frac{s}{\tau u_r}-u^2\Big)(\phi|_{r'}+C)\,\mathrm{d}s\,\mathrm{d}^3u$$
$$=\int_{-\infty}^{\infty}\int_0^b\frac{u_z^2}{u_r}\exp\Big(-\frac{s}{\tau u_r}-u^2\Big)$$
$$\times[\phi+C-\bar{L}(\phi)s+\tfrac{1}{2}\bar{P}(\phi)s^2]\,\mathrm{d}s\,\mathrm{d}^3u$$
$$=-\tau\int_{-\infty}^{\infty}(\phi+C)u_z^2\Big[\exp\Big(-\frac{b}{\tau u_r}\Big)-1\Big] \quad (22)$$
$$\times\exp(-u^2)\,\mathrm{d}^3u$$
$$-\int_{-\infty}^{\infty}\frac{u_z^2}{u_r}G\exp(-u^2)M\,\mathrm{d}^3u$$
$$-\frac{1}{2\tau}\int_{-\infty}^{\infty}\frac{u_z^2}{u_r^2}\exp(-u^2)N$$
$$\times[G(1+\tau\beta u_z)+\tau u_r\bar{L}(Q)]\,\mathrm{d}^3u.$$

It should be noted that the newly obtained integrals also contain the function ϕ in an explicit form. Therefore, equation (21) should be substituted into (22) so many times till the integrals containing ϕ, become less than the prescribed error of calculations; to estimate the remainder terms, the following expression is used

$$\phi=\frac{1}{\pi^{3/2}}(g+2\mathbf{u}\cdot\mathbf{f}) \quad (23)$$

which, as it follows from (6), corresponds to the equilibrium distribution function.

As a result of similar transformations, we get from equations (14)–(16) a homogeneous set of ordinary differential equations for three unknown functions $\kappa(r), f(r), f_z(r)$, as well as for the parameter β [here (16) reduces to an equation with respect to $\kappa = C + (g/\pi^{3/2})$]. They are not given here for their awkwardness. The resulting set of equations is solved by the collocation method. The unknown functions κ, f, f_z are approximated by the tenth-order polynomials. Substitution of the chosen polynomials into the set of differential equations yields a set of homogeneous algebraic equations with respect to the polynomial coefficients. The parameter β is defined from the condition that the determinant of this set equals zero. The values of the functions κ, f, f_z are calculated accurate to the same arbitrary factor which cannot be defined within the framework of the stated problem. However, as it follows from (18), that the macroscopic quantities could be determined, there is no need in determining this factor because the density, velocities and fluxes are expressed only in terms of relations of the appropriate functions. As a result, formulae (18) fully determine the desired hydrodynamic quantities of the problem.

II

Let us consider in detail two limiting cases, whose analysis is much more simple.

First, we shall analyze the limiting vapor flow regime at $\tau\to 0$, i.e. a continuum regime. In the above set of differential equations for the functions $\kappa(r), f(r), f_z(r)$, substitution of integration with respect to the capillary section area by integration with respect to area of the circle with the radius $A\tau$ (A is chosen so that for any integer $n>0$ $T_m(A)\ll\tau^n$), gives

$$\tau\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big(r\frac{\mathrm{d}f_z}{\mathrm{d}r}\Big)+3\beta^2\tau f_z+2\beta\tau\Big(2f+r\frac{\mathrm{d}f}{\mathrm{d}r}\Big)-\beta\pi^{3/2}\kappa=0, \quad (24)$$

$$\tfrac{1}{2}\tau\pi^{3/2}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big(r\frac{\mathrm{d}\kappa}{\mathrm{d}r}\Big)-\tfrac{3}{2}\tau^2\Big(r\frac{\mathrm{d}^3f}{\mathrm{d}r^3}+5\frac{\mathrm{d}^2f}{\mathrm{d}r^2}+\frac{3}{r}\frac{\mathrm{d}f}{\mathrm{d}r}\Big)$$
$$-\tau\beta^2\Big[\pi^{3/2}\kappa-3\beta\tau f_z-\tfrac{3}{2}\tau\Big(2f+r\frac{\mathrm{d}f}{\mathrm{d}r}\Big)\Big]$$
$$-2f-r\frac{\mathrm{d}f}{\mathrm{d}r}-\beta f_z=0, \quad (25)$$

$$\pi^{3/2}\frac{\mathrm{d}\kappa}{\mathrm{d}r}-3\tau\Big(3\frac{\mathrm{d}f}{\mathrm{d}r}+r\frac{\mathrm{d}^2f}{\mathrm{d}r^2}\Big)-\beta\tau\Big(\beta rf+2\frac{\mathrm{d}f_z}{\mathrm{d}r}\Big)=0. \quad (26)$$

From equations (24)–(26), differentiating (26) with respect to r, we have

$$r\frac{\mathrm{d}f}{\mathrm{d}r}+2f+\beta f_z=0. \quad (27)$$

With (27) equation (26) gives

$$\pi^{3/2}\frac{\mathrm{d}\kappa}{\mathrm{d}r}+\beta\tau\frac{\mathrm{d}f_z}{\mathrm{d}r}-\beta^2\tau rf=0. \quad (28)$$

It follows from (24) and (27) that

$$\tau\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big(r\frac{\mathrm{d}f_z}{\mathrm{d}r}\Big)-\beta(\pi^{3/2}\kappa-\beta\tau f_z)=0. \quad (29)$$

Thus, in the present problem equations (27)–(29) with (18) are the main equations used to describe the regime of continuum.

Differentiating (28) and combining the newly obtained equation with (29), we obtain an equation for defining κ

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\kappa}{dr}\right)+\beta^2\kappa = 0. \quad (30)$$

The solution to equation (30) is written in the form

$$\kappa = B\frac{\tau}{\pi^{3/2}}\beta J_0(\beta r) \quad (31)$$

where $J_0(\beta r)$ is the first-kind Bessel function.

Substituting (31) into (29) and solving it with respect to f_z, we have

$$f_z = B[\alpha J_0(\beta r)+W(r)]. \quad (32)$$

Here α and B are the constants of integration, and a particular solution of the nonhomogeneous equation is

$$W(r) = \sum_{s=1}^{\infty} a_{2s}r^{2s}$$

where

$$a_0 = 0, \quad a_{2s} = -a_{2s-2}\left(\frac{\beta}{2s}\right)^2 + (-1)^{s-1}\frac{1}{(2s)^2[(2s-2)!!]^2}\beta^{2s}.$$

Write equation (32) in the form

$$f_z = -B\frac{W(1)}{J_0(\beta)}\left[J_0(\beta r)-\frac{J_0(\beta)}{W(1)}W(r)+d_w\right]. \quad (33)$$

Finally, from (27) and (33) we have

$$rf = B\beta\frac{W(1)}{J_0(\beta)} \times \frac{1}{r}\int_0^r\left[J_0(\beta r)-\frac{J_0(\beta)}{W(1)}W(r)+\mathrm{d}_w\right]r\,dr. \quad (34)$$

Note that equations (31), (33) and (34) have the arbitrary constant B, which cannot be specified. However, it is inessential for the macroscopic values to be obtained, since according to (18), the density, velocities and fluxes are expressed only in terms of the relations of the appropriate functions.

To define β and d_w, it is necessary to consider integral equations (14) and (15) at the points $(1-r) \ll 1$. We introduce a new variable $\eta = (1/\tau)(1-r)$ and the following functions [9]:

$$\tilde{\kappa} = \kappa+H(\eta),$$
$$\tilde{f}_z = f_z+\Phi(\eta), \quad (35)$$
$$\tilde{rf} = rf+\psi(\eta).$$

Here the functions κ, f_z and f are defined from relations (31), (33), (34); the functions $H(\eta)$, $\Phi(\eta)$, $\psi(\eta)$ are correction ones, and at $\eta \to \infty$ they go to zero more rapidly than η^{-n} (n is any positive number).

Expressing the functions κ, f_z and f in terms of a new variable η accurate to the terms of order $\tau\eta$, we have

$$\kappa = B\frac{\tau\beta}{\pi^{3/2}}[J_0(\beta)+\beta\tau\eta J_1(\beta)], \quad (36)$$

$$f_z = -B\frac{W(1)}{J_0(\beta)}\left\{\left[J_1(\beta)+\frac{J_0(\beta)}{\beta W(1)}\left.\frac{dW(x)}{dx}\right|_{x=1}\right] \times \tau\beta\eta+d_w\right\}, \quad (37)$$

$$rf = B\beta\frac{W(1)}{J_0(\beta)}\left\{(1+\tau\eta)\int_0^1\left[J_0(\beta r) - \frac{J_0(\beta)}{W(1)}W(r)\right]r\,dr+\frac{d_w}{2}\right\}. \quad (38)$$

To calculate the integrals on the RHS of equations (14) and (15), use is made of equation (23) and of the method described in [9].

As a result, equation (15), written for the points $(1-r) \ll 1$ with (35)–(38) substituted into it accurate to the terms of order τ, has the form

$$B\frac{\tau\beta}{\pi^{1/2}}J_0(\beta)T_1(\eta)+\frac{2}{\pi^{1/2}}B\beta\frac{W(1)}{J_0(\beta)} \times\left\{-T_2(\eta)+\frac{\tau}{2}[\eta T_0(\eta)+T_1(\eta)-\eta T_2(\eta)+T_3(\eta)]\right\} \times\int_0^1\left[J_0(\beta r)-\frac{J_0(\beta)}{W(1)}W(r)\right]r\,dr$$
$$-\frac{Bd_w\beta}{\pi^{1/2}}\frac{W(1)}{J_0(\beta)}T_2(\eta)-\psi(\eta) -\pi\int_0^\infty G(\eta_0)\,\mathrm{sign}(\eta-\eta_0)T_0(|\eta-\eta_0|)\,d\eta_0$$
$$+\frac{2}{\pi^{1/2}}\int_0^\infty\psi(\eta_0)\left[1+\frac{\tau}{2}(\eta-\eta_0)\right]\left[T_1(|\eta-\eta_0|) -\frac{\tau}{2}(\eta-\eta_0)T_{-1}(|\eta-\eta_0|)\right]d\eta_0 = 0. \quad (39)$$

Let us represent all the desired functions in terms of series in positive ascending powers of τ. We shall write down these series for β, Φ, and d_w:

$$\beta(\tau) = \beta_0+\tau^{n_1}\beta_1+\tau^{n_2}\beta_2+\ldots,$$
$$\Phi(\tau,\eta) = -\frac{B}{\pi^{1/2}}[8\tau^{k_1}\Phi_1(\eta)+4\tau^{k_2}\Phi_2(\eta)+\ldots],$$
$$d_w(\tau) = 2\tau^{l_1}d_{w1}+\tau^{l_2}d_{w2}+\ldots. \quad (40)$$

Substituting (40) into equation (39), we get in the zero approximation

$$\beta_0 = 0. \quad (41)$$

Note, that from (33) and (41) it follows that in such an approximation the axial velocity has a parabolic profile.

Equating the coefficients of τ and letting η approach zero, we obtain the following equation to determine n_1 and β_1:

$$\frac{W(1)}{J_0(\beta_1\tau^{n_1})}\left[\frac{1}{2}-\frac{1}{W(1)}\int_0^1 W(r)r\,dr\right]-\frac{\tau}{\pi^{1/2}} = 0. \quad (42)$$

On the basis of the analytical expressions for $J_0(x)$ and $W(x)$, we obtain from (42)

$$n_1 = \tfrac{1}{2}, \quad \beta_1 = \frac{4}{\pi^{1/4}}. \quad (43)$$

Thus,

$$\beta = \frac{4}{\pi^{1/4}}\tau^{1/2} + o(\tau^{1/2}).$$

Note that assuming $r = 1$ in (36), (38) and taking account of (41), (43), we may show that on the lateral surface the Hertz–Knudsen condition is fulfilled (at least, accurate to the terms of order τ):

$$f = \pi\kappa.$$

From (31), (33) and (34) with account of (43) it follows that

$$\kappa = \frac{B\tau\beta}{\pi^{3/2}}, \qquad f = \frac{B\beta^3}{16}, \tag{44}$$

$$f_z = -\frac{4B\tau}{\pi^{1/2}}\left[2\tau\eta\left(1+\frac{2\tau}{\pi^{1/2}}\right)\left(1-\frac{\tau\eta}{2}\right)+d_w\right].$$

Consider now equation (14) for $(1-r) \ll 1$. Using (44) we get

$$\pi\Phi(\eta) + 8\tau^2 B\left\{T_1(\eta) + \frac{\tau}{2}[\eta T_1(\eta) + 4T_2(\eta) - 2T_0(\eta)]\right\} - 4d_w\tau B\left\{T_0(\eta) + \frac{\tau}{2}[\eta T_0(\eta) + T_1(\eta)]\right\} - \pi^{1/2}\int_0^\infty \Phi(\eta_0)T_{-1}(|\eta-\eta_0|) \times \left[1+\frac{\tau}{2}(\eta-\eta_0)\right]d\eta_0 = 0. \tag{45}$$

From equation (45) it follows that in (40)

$$l_1 = 1, \quad k_1 = l_2 = 2, \quad k_2 = 3.$$

With (40), we obtain from (45) the following equations which allow d_{w1}, d_{w2}, $\Phi_1(\eta)$, $\Phi_2(\eta)$ to be determined:

$$\pi^{1/2}\Phi_1(\eta) = T_1(\eta) - T_0(\eta)d_{w1} + \int_0^\infty \Phi_1(\eta_0)T_{-1}(|\eta-\eta_0|)\,d\eta_0,$$

$$\pi^{1/2}\Phi_2(\eta) = \eta T_1(\eta) + 4T_2(\eta) - 2T_0(\eta) + d_{w1}[\eta T_0(\eta) + T_1(\eta)] - d_{w2}T_0(\eta) + \int_0^\infty [\tfrac{1}{2}\Phi_1(\eta_0)(\eta-\eta_0) + \Phi_2(\eta_0)] \times T_{-1}(|\eta-\eta_0|)\,d\eta_0. \tag{46}$$

Equations (46) coincide with the equations obtained for the Poiseuille flow in [9] where their solutions are given, i.e. expressions for the functions $\Phi_1(\eta)$, $\Phi_2(\eta)$ and the values of d_{w1}, d_{w2}.

Thus, on having obtained β and d_w, expressions (31), (33), (34) determine the solution to the present problem for the conditions of the continuous medium, and the functions H, Φ, ψ allow description of the flow in a thin wall layer.

Let us consider now a system of the integral equations (14)–(16) at $\tau \to \infty$ (a free molecular flow regime). We substitute (23) into the integrals which contain the function ϕ in an explicit form. To simplify the calculation of the integrals, we restrict ourselves to the parabolic dependence of the functions κ, f_z, f on the radius:

$$\kappa = c_0 + b_0 r^2, \quad f_z = c_1 + b_1 r^2, \quad f = c_2 + b_2 r^2.$$

Using asymptotic expansions of the functions $T_1(b/\tau)$, $T_2(b/\tau)$, $T_3(b/\tau)$ at $\tau \to \infty$ [8], we obtain the following algebraic equations to determine the coefficients c_i, b_i:

$$\begin{aligned}
\beta\pi^2 c_0 + \tfrac{1}{3}\beta\pi^2 b_0 + 2c_1 - \beta c_2 - \tfrac{1}{2}\beta b_2 &= 0,\\
\beta\pi^2 c_0 - \beta\pi^2 b_0 - 8b_1 - 4\beta c_2 &= 0,\\
\pi c_0 + \beta c_1 + \tfrac{1}{3}\beta b_1 &= 0,\\
4\pi b_0 - \beta c_1 + \beta b_1 &= 0,\\
\beta c_1 + 2c_2 &= 0,\\
\beta b_1 + 4b_2 &= 0.
\end{aligned} \tag{47}$$

The solution of the set (47) is

$$\frac{b_0}{c_0} = -0.33, \quad \frac{b_1}{c_1} = -0.22, \quad \frac{b_2}{c_2} = -0.11,$$

$$\frac{c_1}{c_0} = -3.55, \quad \frac{c_2}{c_1} = -0.485, \quad \beta = 0.97.$$

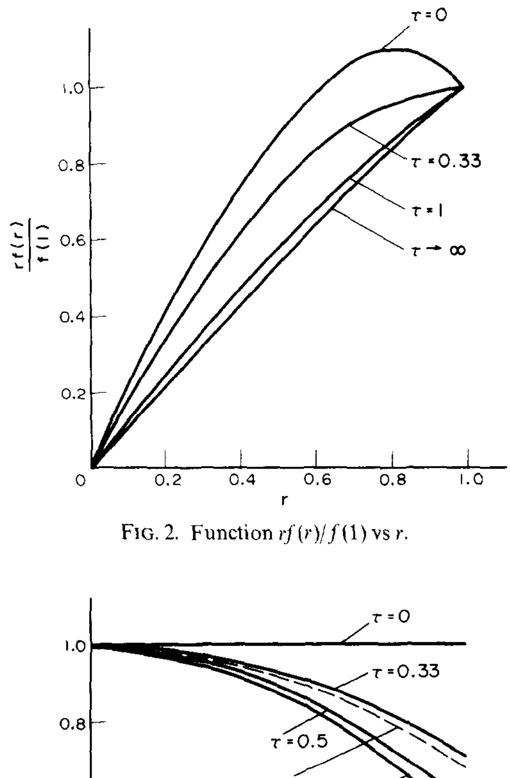

FIG. 2. Function $rf(r)/f(1)$ vs r.

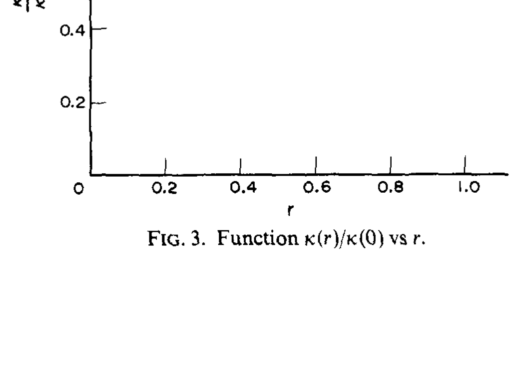

FIG. 3. Function $\kappa(r)/\kappa(0)$ vs r.

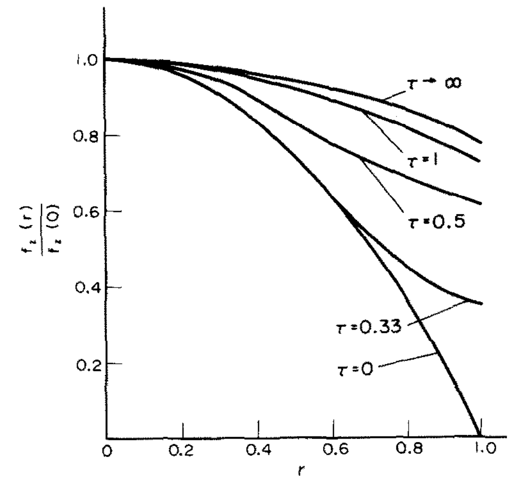

FIG. 4. Function $f_z(r)/f_z(0)$ vs r.

Table 1. Dependence of $\bar{\kappa}/\bar{f}_z$, $\bar{\kappa}/f(1)$ and β on τ

τ	0	0.33	0.50	1.0	2.0	∞
$-\bar{\kappa}/\bar{f}_z$	0	0.133	0.171	0.218	0.248	0.270
$\bar{\kappa}/f(1)$	0.318	0.375	0.442	0.501	0.543	0.556
β	0	0.708	0.773	0.862	0.912	0.970

Hence,

$$\kappa = c_0(1-0.33r^2), \quad f_z = c_1(1-0.22r^2),$$
$$f = c_2(1-0.11r^2).$$

It should be noted that according to the solution for a free-molecular regime, obtained by an alternate means [10], $\beta = 1$.

The Table summarizes the values of $\bar{\kappa}/\bar{f}_z$, $\bar{\kappa}/f(1)$ for different τ, which permit the cross-section averaged flow in the z-direction as well as the flux on the lateral surface to be determined by formulae (18). Figures 2–4 present the distributions of the functions κ, rf and f_z over the radius, necessary for determining the profiles and velocities (in this case the approximation used at $\tau \to \infty$ for $\kappa(r)$ turned out to be insufficiently accurate).

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CINETIQUE DU TRANSFERT DE MASSE DANS UN CAPILLAIRE AVEC EVAPORATION SUR LA SURFACE INTERNE

Résumé—On donne la formulation mathématique et la méthode de résolution du problème de la cinétique du transfert massique dans un capillaire avec évaporation à la surface interne. Une équation linéarisée BKG est résolue la fonction de distribution étant choisie de façon que les conditions limites soient satisfaites pour l'équation originale divisée en deux équations pour les fonctions dépendant respectivement de z et de r. Des expressions sont obtenues pour les distributions des grandeurs macroscopiques caractéristiques de l'écoulement de vapeur. Dans une seconde partie de l'article, sur la base de la technique générale donnée, on étudie analytiquement le transfert massique pour les cas limites d'un milieu continu et d'un régime moléculaire libre. Les tables et les figures présentent les valeurs des grandeurs qui concernent l'écoulement moyen dans la section droite selon la direction axiale et l'écoulement sur la surface latérale et les profils de masse spécifique et de vitesse pour différents régimes de raréfaction.

DIE KINETIK DER STOFFÜBERTRAGUNG IN EINER KAPILLARE MIT VERDAMPFUNG AN DER INNEREN OBERFLÄCHE

Zusammenfassung—Die Arbeit beschreibt die mathematische Formulierung und die Lösungsmethode des Problems der Stoffübertragungskinetik in einer Kapillare mit Berücksichtigung der Verdampfung an der Innenfläche. Es wird die linearisierte Gleichung eines BKG-Modells gelöst, wobei die Distributionsfunktion derart gewählt wurde, daß sowohl die Randbedingungen erfüllt als auch die ursprüngliche Gleichung in zwei von z bzw. r abhängige Gleichungen aufgeteilt werden konnte. Es werden Ausdrücke für die Verteilung makroskopischer den Dampfstrom charakterisierende Größen erhalten. Im zweiten Teil der Arbeit wird auf der Grundlage der allgemein gegebenen Technik der Stofftransport in einer Kapillare analytisch für die Grenzfälle des kontinuierlichen Mediums und der freien Molekular-Strömung behandelt. Die Tabellen und Bilder geben Größen an, nach denen sich sowohl der über den Querschnitt gemittelte axiale Fluß als auch der Fluß an einer seitlichen Oberfläche sowie die Dichte- und Geschwindigkeitsprofile für verschiedene Verdünnungszustände ermitteln lassen.

КИНЕТИКА МАССОПЕРЕНОСА В КАПИЛЛЯРЕ ПРИ ИСПАРЕНИИ С ВНУТРЕННЕЙ ПОВЕРХНОСТИ

Аннотация — В работе рассматриваются математическая постановка и метод решения задачи о кинетике массопереноса в капилляре при испарении с внутренней поверхности. Решается линеаризованное модельное уравнение БГК. При этом функции распределения выбирается таким, чтобы удовлетворялись граничные условия, а также имело место разделение исходного уравнения на два уравнения для функций, зависящих от z и от r соответственно. Получены выражения для распределений макроскопических величин, характеризующих течение пара.

Во второй части работы, исходя из приведенной общей методики, аналитически исследуется массоперенос в капилляре для предельных случаев сплошной среды и свободномолекулярного режима. В таблице и на рисунках приводятся значения величин, позволяющие определить средний по сечению поток в аксиальном направлении и поток на боковой поверхности, а также профили плотности и скоростей для различных режимов разрежения.